TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 303, Number 1, September 1987

SCALAR CURVATURE AND WARPED PRODUCTS OF RIEMANN MANIFOLDS

F. DOBARRO AND E. LAMI DOZO

ABSTRACT. We establish the relationship between the scalar curvature of a warped product $M \times_f N$ of Riemann manifolds and those ones of M and N. Then we search for weights f to obtain constant scalar curvature on $M \times_f N$ when M is compact.

1. Introduction. Let $M = (M_m, g)$ and $N = (N_n, h)$ be two Riemann manifolds. For $f \in C^\infty(M)$, $f > 0$ on M, we consider the *warped product* $M \times_f N = ((M \times N)_{m+n}, g + f^2 h)$ and show the relationship between the scalar curvatures R on M, H on N and $\tilde{R}$ on $M \times_f N$. This relationship is a nonlinear partial differential equation satisfied by a power of the *weight* f. In the case M is compact and connected, we obtain a geometric interpretation of the principal eigenfunction and eigenvalue of the canonical elliptic operator $-\Delta + R/2$, where Δ denotes the laplacian on M. Finally we consider the question of finding a weight f such that $M \times_f N$ has constant scalar curvature. This question is equivalent to find a positive solution to a nonlinear eigenvalue problem.

The notion of warped product $M \times_f N$ generalizes that of a surface of revolution. It was introduced in [**B-O**] for studying manifolds of negative curvature (cf. [**Z**] for other applications). The Riemann metric $\tilde{g} = g + f^2 h$ on $M \times_f N$ is defined for pairs of vector fields $\tilde{X}, \tilde{Y}$ on $M \times N$ by

$$\tilde{g}(\tilde{X}, \tilde{Y}) = g(\pi_* \tilde{X}, \pi_* \tilde{Y}) + f^2(\pi(\cdot)) h(\omega_* \tilde{X}, \omega_* \tilde{Y})$$

where π and ω are the canonical projections over M and N respectively.

We denote by Δ_g or Δ the laplacian (or Laplace-Beltrami) operator on (M_m, g) with local expression $\Delta_g u = \nabla^i \nabla_i u = |g|^{-1/2} \partial_i (g^{ij} |g|^{1/2} \partial_j u)$, for $u \in C^2(M)$ (cf. [**Au1, B-G-M**]). Thus $\Delta u = u''$ for a real valued function u on $M = \mathbf{R}$.

2. The equation. Given a metric $g' = kg$ with $k \in C^\infty(M)$, $k > 0$ on M, g' is said to be *conformal* to g. It is known that the scalar curvature R' on (M_m, g') is related to R, the one on (M_m, g), by the Yamabe equation

$$\text{(Ya)} \qquad -\frac{4(m-1)}{m-2} \Delta_g u + Ru = R' u^{(m+2)/(m-2)}$$

where $k = u^{4/(m-2)}$, whenever $m \geq 3$ [**4, Au1**, p. 126].

Received by the editors July 15, 1986.

1980 *Mathematics Subject Classification*. Primary 58G30, 53C25, 58C40; Secondary 47H12, 47H15.

Key words and phrases. Nonlinear eigenvalue problem, principal eigenvalue and eigenfunction, bifurcation from infinity.

If we consider now the case of a warped product we have

THEOREM 2.1. *Let R, H and $\tilde{R}$ denote the scalar curvature on M, N and $M \times_f N$ respectively. Then the following equality holds:*

$$-\frac{4n}{n+1}\Delta_g u + Ru + Hu^{(n-3)/(n+1)} = \tilde{R}u \tag{2.1}$$

where $u = f^{(n+1)/2}$.

PROOF. We write $\tilde{g} \equiv g + f^2 h = f^2(f^{-2}g + h)$, so $\tilde{g}$ is conformal to $\approx{g} = f^{-2}g + h$ on $M \times N$ and $f^{-2}g$ is conformal to g on M.

Supposing $m \geq 3$, we apply (Ya) in M to obtain that f satisfies

$$-\frac{4(m-1)}{m-2}\Delta_g \eta + R\eta = \eta^{(m+2)/(m-2)}\hat{R} \tag{2.2}$$

with $\eta^{4/(m-2)} = f^{-2}$ and where $\hat{R}$ denotes the scalar curvature on $(M_m, f^{-2}g)$. As $m + n \geq 3$, we use (Ya) in $M \times N$. Hence f also satisfies

$$-\frac{4(m+n-1)}{m+n-2}\Delta_{\approx{g}}\psi + \approx{R}\psi = \tilde{R}\psi^{(m+n+2)/(m+n-2)} \tag{2.3}$$

with $\psi^{4/(m+n-2)} = f^2$, where $\approx{R}$ denotes the scalar curvature on $((M \times N)_{m+n}, \approx{g})$ and $\Delta_{\approx{g}}$ the corresponding laplacian.

From $\psi \in C^\infty(M)$ we deduce that $\Delta_{\approx{g}}\psi \equiv \Delta_{f^{-2}g+h}\psi = \Delta_{f^{-2}g}\psi$. Working in local coordinates

$$\Delta_{f^{-2}g}\psi = |f^{-2}g|^{-1/2}\partial_i[(f^{-2}g)^{ij}|f^{-2}g|^{1/2}\partial_j\psi]$$

with $|f^{-2}g| = \det(f^{-2}g_{ij}) = f^{-2m}|g|$ and $(f^{-2}g)^{ij} = f^2 g^{ij}$. Hence

$$\begin{aligned}\Delta_{f^{-2}g}\psi &= \eta^{-2m/(m-2)}|g|^{-1/2}\partial_i[\eta^2 g^{ij}|g|^{1/2}\partial_j\psi] \\ &= [\eta\Delta_g\psi + 2g^{ij}\partial_i\eta\partial_j\psi]\eta^{-(m+2)/(m-2)}.\end{aligned} \tag{2.4}$$

On the other hand

$$\Delta_g(\eta\psi) = \eta\Delta_g\psi + \psi\Delta_g\eta + 2g^{ij}\partial_i\eta\partial_j\psi$$

and from (2.4) we get

$$\eta^{(m+2)/(m-2)}\Delta_{\approx{g}}\psi = \Delta_g(\eta\psi) - \psi\Delta_g\eta. \tag{2.5}$$

We have that $\approx{R} = \hat{R} + H$, because we consider the usual product of $(M_m, f^{-2}g)$ by (N_n, h). Using this in (2.3), multiplying by $\eta^{(m+2)/(m-2)}$, we obtain from (2.5)

$$\begin{aligned}-\frac{4(m+n-1)}{m+n-2}(\Delta_g(\eta\psi) - \psi\Delta_g\eta) + \eta^{(m+2)/(m-2)}(\hat{R} + H)\psi \\ = \tilde{R}\psi^{(m+n+2)/(m+n-2)}\eta^{(m+2)/(m-2)}.\end{aligned}$$

From (2.2) we arrive at

$$\begin{aligned}-\frac{4n}{(m+n-2)(m-2)}\psi\Delta_g\eta - \frac{4(m+n-1)}{(m+n-2)}\Delta_g(\eta\psi) + R\eta\psi + H\eta^{(m+2)/(m-2)}\psi \\ = \tilde{R}\psi\eta\psi^{4/(m+n-2)}\eta^{4/(m-2)}.\end{aligned}$$

Recalling that $\psi^{4/(m+n-2)}\eta^{4/(m-2)} = 1$, denoting $u = f^{(n+1)/2}$, replacing in terms of u in this last equality, then multiplying by $u^{1-n/(n+1)}$, we obtain

$$(2.6)\quad \begin{cases} -\dfrac{4n}{(m+n-2)(m-2)}u^{(m+n-1)/(n+1)}\Delta_g u^{-(m-2)/(n+1)} \\ -\dfrac{4(m+n-1)}{m+n-2}u^{1/(n+1)}\Delta_g u^{n/(n+1)} + Ru + Hu^{(n-3)/(n+1)} = \tilde{R}u. \end{cases}$$

For any $\alpha \neq 0$, $v \in C^\infty(M)$, $v > 0$ on M satisfies

$$(2.7)\qquad \Delta_g v^\alpha = \alpha(\alpha-1)v^{\alpha-2}\nabla^i v\nabla_i v + \alpha v^{\alpha-1}\Delta_g v.$$

Choosing first $\alpha = n/(n+1)$, then $\alpha = -(m-2)/(n+1)$ in (2.7) (for $v = u$) we obtain from (2.6) the desired result (2.1).

REMARK. Formula (2.1) holds even for $m = 1, 2$. For $m = 1$, it is easily deduced from a similar formula in [**E-1**]. For $m = 2$, the proof is similar but Yamabe equation is for the conformal change $g' = e^u g$:

$$(\mathrm{Ya})_2 \qquad -\Delta_g u + R = R'e^u$$

(cf. [**M**, **Au1**, p. 119]). (2.1) could also be deduced from a formula in [**B-O**, p. 26] but this formula is a consequence of an unwritten 15-term calculation and besides our method of proof is different.

3. Constant scalar curvature. Let the scalar curvatures R of M and H of N be fixed, we look for a weight $f \in C^\infty(M)$, $f > 0$, on M such that the warped product $M \times_f N$ has constant scalar curvature $\tilde{R}$ and then which constants are attained. Taking account of $R \in C^\infty(M)$, $u = f^{(n+1)/2} \in C^\infty(M)$ and $H \in C^\infty(N)$, it follows easily from Theorem 2.1 that for $\tilde{R}$ to be a constant λ, it is necessary that N have constant scalar curvature, still denoted H.

The simplest case is $H = 0$.

THEOREM 3.1. *Let M be compact and connected. Suppose N of zero scalar curvature. Then there exists a weight f such that the scalar curvature $\tilde{R}$ on $M \times_f N$ is a constant λ_1. f is unique up to a positive multiplicative constant, λ_1 is unique and is given by*

$$\lambda_1 = \inf\left\{\int_M \left(\frac{4n}{n+1}|\nabla v|^2 + Rv^2\right) dV; v \in H^1(M), \int_M v^2\, dV = 1\right\}$$

where $H^1(M) = \{v \in L^2(M); |\nabla v|^2 \equiv \nabla^i v\nabla_i v \in L^1(M)\}$ is the Sobolev space of order 1.

PROOF. From (2.1) we deduce that we search $\lambda \in \mathbf{R}$ and $u \in C^\infty(M)$, $u > 0$ on M such that

$$(3.1)\qquad Lu = \lambda u \quad \text{on } M$$

where $Lu = -4n\Delta_g u/(n+1) + Ru$.

It is well known that this linear eigenvalue problem on M compact and connected has only one nonnegative solution u_1 with $\max_M u_1 = 1$; cf. [**Au1**, p. 137] (in fact $u_1 > 0$ on M) u_1 is the so-called *principal eigenfunction* of the elliptic operator L. The corresponding eigenvalue λ_1 is simple and is called the *principal eigenvalue.*

Hence $f = u_1^{2/(n+1)} > 0$ on M is the weight we are searching. Any other solution is of the form rf, $r \in \mathbf{R}_0^+$, because λ_1 is simple. The formula for λ_1 is classical.

The case of warping M with a circle, i.e. $M \times_f S^1$ gives a geometric interpretation of the principal eigenfunction u_1 of $-\Delta + R/2$ and its corresponding eigenvalue λ_1, which in the special case of M 2-dimensional can be expressed in terms of the classical Gaussian curvature on M.

COROLLARY. *Given a compact, connected $M = (M_2, g)$ with Gaussian curvature K and Laplace-Beltrami operator Δ, then the principal eigenfunction u_1 of the canonical elliptic operator $-\Delta + K$ holds the property that the warped product $M \times_{u_1} S^1$ has a scalar curvature constantly equal to the principal eigenvalue λ_1 of $-\Delta + K$.*

PROOF. Take $n = 1$ in (2.1). Recall that $H = 0$ in S^1 and $R/2 = K$ in a 2-dimensional manifold.

The case $H \in \mathbf{R}$, $H < 0$ is similar to $H = 0$ because the half-line $\{(\lambda_1, ru_1), r > 0\}$ is deformed into a curve $\{(\lambda, f(\lambda));\ \lambda < \lambda_1\}$:

THEOREM 3.2. *Let $M = (M_m, g)$ be compact and connected. Suppose $N = (N_n, h)$ of constant negative scalar curvature H and assume $n \geq 3$. Let λ_1 denote the principal eigenvalue of $-4n\Delta_g/(n+1) + R$. Then for each $\lambda < \lambda_1$ there exists a unique weight $f = f(\lambda)$ such that $M \times_f N$ has constant scalar curvature λ. No constant $\geq \lambda_1$ may be curvature of $M \times_f N$ for any weight f.*

PROOF. We look for positive solutions u in $C^\infty(M)$ of

$$Lu + Hu^\alpha = \lambda u \quad \text{on } M \tag{3.2}$$

where $Lu = -4n\Delta_g u/(n+1) + Ru$, $0 \leq \alpha = (n-3)/(n+1) < 1$ (cf. (2.1)).

Let us still denote u_1 the positive eigenfunction of $Lu = \lambda u$ but with L^2-norm 1, i.e. $\int_M u_1^2\, dV = 1$. If u is a positive solution of (3.2), then multiplying (3.2) by u_1 and integrating by parts (L is selfadjoint), we obtain

$$\lambda_1 \int_M uu_1\, dV + H \int_M u^\alpha u_1\, dV = \lambda \int_M uu_1\, dV.$$

Hence

$$(\lambda - \lambda_1) \int_M uu_1\, dV = H \int_M u^\alpha u_1\, dV. \tag{3.3}$$

Then $H < 0$ necessarily gives $\lambda < \lambda_1$.

Let us fix $\lambda < \lambda_1$. As $0 \leq \alpha < 1$, we have that

$$(L - \lambda I)\underline{t}u_1 \leq |H|\underline{t}^\alpha u_1^\alpha$$

for $\underline{t} \in \mathbf{R}^+$ small enough, so $\underline{t}u_1$ is a subsolution of (3.2). Also

$$(L - \lambda I)\bar{t}u_1 \geq |H|\bar{t}^\alpha u_1^\alpha$$

for $\bar{r} \in \mathbf{R}^+$ big enough, so we have a supersolution $\bar{t}u_1 \geq \underline{t}u_1$.

The operator $(L - \lambda I)\colon C^{2,\beta}(M) \to C^\beta(M)$ is an isomorphism and its inverse $(L - \lambda I)^{-1}$ is continuous for the C^0-norm on $C^\beta(M)$ and the $C^{1,\beta}$-norm on $C^{2,\beta}(M)$. Besides $(L - \lambda I)^{-1}$ is strongly positive, i.e. $w \in C^\beta(M)$, $w \geq 0$, $w \not\equiv 0$, implies $(L - \lambda I)^{-1} w > 0$ on M. So it extends uniquely to a compact map, still denoted

$(L-\lambda I)^{-1}$ from $C(M)$ into $C(M)$, which is still strongly positive (cf. [**Am**]). For each $w \in C(M)$, $(L-\lambda I)^{-1}w$ is a weak solution of $Lu - \lambda u = w$.

The nonlinear compact and order preserving operator $v \to (L-\lambda I)^{-1}(|H||v|^{\alpha})$: $C(M)^+ \to C(M)^+$ leaves invariant the order interval $[\underline{t}u_1, \bar{t}u_1] \subset C(M)$, so it has a fixed point $u \in C(M)$, with $0 < \underline{t}u_1 \leq u \leq \bar{t}u_1$, i.e. $u = (L-\lambda I)^{-1}(|H||u|^{\alpha})$, hence by a classical bootstrap argument $u \in C^{\infty}(M)$; in particular u is a classical positive solution of (3.2) (cf. [**Am**] in the Neumann case for details). Finally the nonlinearity $|H|t^{\alpha}$ in (3.2) is such that t^{α}/t is strictly decreasing in $t > 0$; hence u is unique [**L**, **Bere**]. As $f = u^{2/(n+1)}$ we obtain the uniqueness.

The situation seems more complicated when $H > 0$.

THEOREM 3.3. *Let $M = (M_m, g)$ be compact and connected. Suppose $N = (N_n, h)$ of constant positive curvature H and assume $n \geq 3$. Let λ_1 denote the principal eigenvalue of $-4n\Delta_g/(n+1) + R$. Then for each λ in some interval $(\lambda_1, \lambda_1 + \delta)$ there exists a weight $f = f(\lambda)$ such that $M \times_f N$ has λ as scalar curvature. No constant $\leq \lambda_1$ may be curvature of $M \times_f N$ for any f.*

PROOF. If $u > 0$ on M is a solution of (3.2) with $H > 0$, then (3.3) gives that $\lambda > \lambda_1$ is a necessary condition.

Denote $\|u\| = \max_M |u|$, $v = u/\|u\|^2$ for $u \neq 0$, i.e. $u = v/\|v\|^2$. We will obtain solutions as a bifurcation from infinity near $\lambda = \lambda_1$ (cf. [**R-2**]). Multiplying our equation (3.2) by $1/\|u\|^2$, replacing by v, we are reduced to finding positive solutions of

$$Lv + av = \lambda v - H\|v\|^{2(1-\alpha)}v^{\alpha} + av \tag{3.4}$$

where $\lambda \in \mathbf{R}$ is a parameter and $a \in \mathbf{R}^+$ is a fixed number chosen big enough so that the operator $(L + aI)\colon C^{2,\beta}(M) \to C^{\beta}(M)$ is an isomorphism with positive inverse. Hence, as in the preceding proof, $(L+aI)^{-1}\colon C(M) \to C(M)$ is a linear compact strongly prositive operator such that $(L+aI)^{-1}v > 0$ on M if $v \geq 0$, $v \not\equiv 0$ on M.

We now search for pairs $(\lambda, v) \in \mathbf{R} \times C(M)$ with $v > 0$ on M, solutions of

$$v = \mu(L+aI)^{-1}v - H(L+aI)^{-1}f(v) \tag{3.5}$$

where $\mu = \lambda + a$, $H > 0$, and $f(v) = \|v\|^{2(1-\alpha)}|v|^{\alpha}$. We easily see that $H(L+aI)^{-1}f(v) = o(\|v\|)$ in $C(M)$ for $\|v\|$ near 0. Moreover from $Lu_1 = \lambda_1 u_1$, it follows that $\mu_1 = \lambda_1 + a$ is a simple characteristic value of the compact map $(L+aI)^{-1}$. Then by the Rabinowitz bifurcation theorem [**R-1**], there exists a maximal connected closed subset $\mathcal{C}_{\mu_1}$ of

$$\mathcal{S} = \mathrm{adh}_{\mathbf{R}\times C(M)}\{(\lambda, v) \text{ solutions of (3.5) with } v \neq 0\}$$

such that $(\mu_1, 0) \in \mathcal{C}_{\mu_1}$ and either

(i) $\mathcal{C}_{\mu_1}$ is bounded in $\mathbf{R} \times C(M)$ or

(ii) $\mathcal{C}_{\mu_1}$ meets $(\hat{\mu}, 0)$ with $\mu_1 \neq \hat{\mu} \in \{\lambda + a; \lambda \text{ eigenvalue of } L\}$.

We write $v = \gamma u_1 + w$, with $\gamma = v(x_o)$, where $u_1(x_o) = \|u_1\|$ and $w(x_o) = 0$. For $\xi > 0$, $0 < \eta < 1$, the open sets in $\mathbf{R} \times C(M)$: $K^+_{\xi,\eta}, K^-_{\xi,\eta}$ defined by

$$K^{\pm}_{\xi,\eta} = \{(\mu, v) \in \mathbf{R} \times C(M); |\mu - \mu_1| < \xi, \pm\gamma < \eta\|v\|\} \tag{3.6}$$

satisfy both $K^{\pm}_{\xi,\eta} \cap \mathcal{C}_{\mu_1} \neq \varnothing$ for ξ small enough as in Theorem 1.25 in [**R-1**]. Consequently the maximal connected closed subset (or continuum) of $\mathcal{C}_{\mu_1}$ contained in $\{(\mu, \gamma u_1 + w); \gamma > 0\} \cup \{(\mu_1, 0)\}$ is nontrivial; we denote it as $\mathcal{C}^+_{\mu_1}$. Besides from $H(L+aI)^{-1}f(v) = o(\|v\|)$ we deduce that $\|w\| = o(\gamma)$ near $\gamma = 0$, so $v = \gamma u_1 + w \in \mathcal{C}^+_{\mu_1}$ is (strictly) positive on M for $\gamma > 0$ small enough.

We have then obtained a "branch" of $C(M)$-solutions (μ, v), $v > 0$ on M of (3.5). Going back to $u = v/\|v\|^2$, $\lambda = \mu - a$, we have weak $C(M)$-solutions of (3.2): (λ, u) for λ near λ_1, $u > 0$ on M, $\{\|u\|\}$ unbounded. By a regularity argument, each u is a C^∞ classical solution and by the necessary condition we have $\lambda > \lambda_1$.

REMARK 1. The nonlinearity $v \to (L+aI)^{-1}f(v) = \|v\|^{2(1-\alpha)}(L+aI)^{-1}(|v|^\alpha)$ is not always differentiable for $\|v\|$ near 0, so we cannot directly apply the results in [**R-1**, **2**].

REMARK 2. The operator $F(\mu, v) = (L+aI)^{-1}(\mu v - Hf(v))$, $H > 0$, generally transforms a positive v into a nonpositive $F(\mu, v)$, so some known results on maps of cones into cones [**L**, **T**] do not apply. More precisely, when N is 3-dimensional, (3.2) can be written as

$$(-3\Delta_g + RI - \lambda I)u = -H \tag{3.7}$$

and Theorem 3.3 is a consequence of the antimaximum principle (cf. [**C-P**]) which says that $\lambda > \lambda_1$ and near λ_1 implies that negative data $-H$ on M gives a positive solution u, on the contrary of the maximum principle.

Let us denote

$$\mathcal{C}_\infty = \{(\lambda, u); u \geq 0, u \neq 0, (\lambda + a, u/\|u\|^2) \in \mathcal{C}^+_{\mu_1}\} \subset \mathbf{R} \times C(M)^+,$$

the nonnegative weak $C(M)$-solutions bifurcating from infinity. We know now that $H > 0$ fixed implies that $u > 0$ for $\lambda > \lambda_1$, λ near λ_1. We can say more on $\|u\|$ in $\mathcal{C}_\infty$.

THEOREM 3.4. *Let M, N and λ_1 be as in Theorem* 3.3. *For any* $0 < \varepsilon < A$, *the set* $\{(\lambda, u) \in \mathcal{C}_\infty;\ \lambda_1 + \varepsilon \leq \lambda \leq \lambda_1 + A\}$ *is bounded in* $\mathbf{R} \times C(M)$.

PROOF. Suppose not. Then there exists a sequence $(\lambda_n, u_n) \in \mathcal{C}_\infty$ with $\lambda_n \in [\lambda_1 + \varepsilon, \lambda_1 + A]$, $u_n \geq 0$ on M, $u_n \neq 0$, $\lim_n \lambda_n = \lambda$ and $\lim_n \|u_n\| = +\infty$, which satisfies

$$u_n/\|u_n\| = (\lambda_n + a)(L+aI)^{-1}(u_n/\|u_n\|) - H(L+aI)^{-1}(u_n^\alpha/\|u_n\|).$$

$(L+aI)^{-1}\colon C(M) \to C(M)$ being compact we may suppose, up to a subsequence, that $\lim_n (L+aI)^{-1}(u_n/\|u_n\|) = \tilde{u}$ in $C(M)$. From $\lim_n(u_n^\alpha/\|u_n\|) = 0$ in $C(M)$, we obtain that $u_n/\|u_n\|$ tends to some u in $C(M)$, $u \geq 0$, $\|u\| = 1$, so we have

$$u = (\lambda + a)(L+aI)^{-1}u$$

i.e., by regularity properties, $-4n\Delta u/(n+1) + Ru = \lambda u$, $v \geq 0$, $u \neq 0$, hence $u = u_1$ and $\lambda = \lambda_1$ by uniqueness. This contradicts $\lambda \geq \lambda_1 + \varepsilon$.

REMARK. If the set $\mathcal{C}_\infty$ meets $(\hat{\mu}, 0) \in \mathbf{R} \times C(M)$ so does $\mathcal{C}^+_{\mu_1}$. Then $\hat{\mu} \neq \mu_1$ and $\hat{\mu}$ is necessarily a characteristic value of $(L+aI)^{-1}$ (cf. [**R-1**]). Hence $\hat{\mu} = \hat{\lambda} + a$, $\hat{\lambda} > \lambda_1$, $\hat{\lambda}$ an eigenvalue of $-4n\Delta/(n+1) + R$. We would then have a sequence $(\mu_k, v_k) \in \mathcal{C}^+_{\mu_1}$, such that $\lim_k \mu_k = \hat{\mu}$, $\lim_k v_k = \lim_k u_k/\|u_k\|^2 = 0$ in $C(M)$, $v_k \neq 0$, i.e. $(\lambda_k, u_k) \in \mathcal{C}_\infty$, with $\lambda_k = \mu_k - a \geq \lambda_1 + \varepsilon$, $\{\lambda_k\}$ bounded and $\lim_k \|u_k\| = +\infty$, contradicting Theorem 3.4. Then $\mathcal{C}_\infty$ never meets some $(\lambda, 0)$.

For $n = 3$, $H > 0$ we are able to characterize the case that for any $\lambda > \lambda_1$ there is a $u(\lambda) > 0$ on M with $(\lambda, u(\lambda)) \in \mathcal{C}_\infty$.

THEOREM 3.5. *Let M, N and λ_1 be as in Theorem 3.3 with $\dim N = 3$. Then any constant $\lambda \in (\lambda_1, \infty)$ is scalar curvature of $M \times_f N$ for some weight f if and only if the scalar curvature R on M is constant.*

PROOF. If R is constant and $\lambda > \lambda_1$, $u(\lambda) = H/(\lambda - \lambda_1)$ is a solution of (3.7), hence $f = u^{1/2}$ is a constant weight which gives λ as scalar curvature on $M \times_f N$.

Conversely, suppose that for each $\lambda > \lambda_1$ there corresponds a weight f i.e. a solution u of (3.7). This means for $\lambda = \lambda_k$ $(k > 1)$ an eigenvalue of the selfadjoint operator $-3\Delta_g + R$, that the second member in equality (3.7) is orthogonal to the corresponding eigenspace. Hence H is orthogonal to all eigenfunctions except the principal one u_1, so $H = tu_1$ with $t > 0$ by the completeness of an orthonormal system of eigenfunctions, i.e. u_1 is constant and (3.7) with $u = u_1$ gives R constant.

Ejiri proves in [**E-2**] that there exist countable immersions of $S^1 \times S^n$ into S^{n+2} such that $S^1 \times S^n$ is a warped product of constant scalar curvature $n(n+1)$ with respect to the induced metric.

On one hand, if we consider equation (2.1) in $S^1 \times S^n$, we have $R = 0$ on S^1 and $H = n(n-1)$ on S^n with the usual metric. So [**E-2**] gives a countable number of positive solutions of

$$-\frac{4n}{n+1}u'' + Hu^{(n-3)/(n+1)} = n(n+1)u. \tag{3.8}$$

In particular for $n = 3$, this equation becomes

$$-3u'' - 12u = -H \tag{3.9}$$

with $H = 6$.

On the other hand, equation (3.7) reduces on $S^1 \times S^3$ to

$$-3u'' - \lambda u = -H. \tag{3.10}$$

This last equation has a unique solution $u = u(\lambda)$ for $\lambda > \lambda_1$, λ not an eigenvalue. But $\lambda = 12$ is an eigenvalue of (-3) times the laplacian on S^1, the eigenvalues being $\{\lambda_k = 3(k-1)^2, k = 1, 2, \dots\}$, so the countable solutions of (3.9) given in [**E-2**] appear for $\lambda = \lambda_3$. But then we have uncountable positive solutions of (3.10) for $\lambda = \lambda_k$, $k > 1$, they are

$$v = H/\lambda_k + tu_k \tag{3.11}$$

where $-3u_k'' = \lambda_k u_k$ and $|t|$ small enough so that v is positive. We have proved

THEOREM 3.6. *Let M, N and λ_1 be as in Theorem 3.3 with $\dim M = 1$ and $\dim N = 3$. Given an eigenvalue λ_k, $k > 1$, of $-3\Delta_g$ there exist uncountable weights f such that $M \times_f N$ has λ_k as scalar curvature.*

REMARK. The $u = f^2$ from this theorem are secondary bifurcations of the branch $\{H/\lambda\}$ of solutions considered up to now, at points $\lambda = \lambda_k$. In the case $M = S^1$ and $N = S^n$ with $n > 3$, the curvature on $S^1 \times_f S^n$ given in [**E-2**] is $n(n+1)$ with $u = f^{(n+1)/2}$ a solution of (3.8). This constant $n(n+1)$ is far from the scalar curvatures obtained from Theorem 3.3 which are near zero, i.e. our solutions of (2.1) and those in [**E-2**] on $S^1 \times S^n$ seem to be of different type.

Theorem 3.2 still holds for N of dimension 2 with a different proof inspired from [**C-R-T**].

REFERENCES

[**Am**] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), 620-709.

[**Au1**] T. Aubin, *Nonlinear analysis on manifolds. Monge-Ampère equations*, A Series of Comprehensive Studies in Mathematics 252, Springer, Berlin.

[**Au2**] ——, *Métriques Riemanniennes et courbure*, J. Differential Geom. **4** (1970), 383–424.

[**Bere**] H. Berestycki, *Le nombre de solutions de certains problèmes semi-linéaires elliptiques*, J. Funct. Anal. **40** (1981), 1-29.

[**B-G-M**] M. Berger, P. Gauduchon, and E. Mazet, *Le spectre d'une variété Riemannienne*, Lecture Notes in Math., vol. 194, Springer, 1971.

[**B-O**] R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. **145** (1969), 1-49.

[**C-P**] Ph. Clément and L. A. Peletier, *An anti-maximum principle for second-order elliptic operators*, J. Differential Equations **34** (1979), 218–229.

[**C-R-T**] M. G. Crandall, P. H. Rabinowitz, and L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, MRC Report 1680, 1976.

[**E-1**] N. Ejiri, *A negative answer to a conjecture of conformal transformations of Riemann manifolds*, J. Math. Soc. Japan **33** (1981), 261–266.

[**E-2**] ——, *Some compact hypersurfaces of constant scalar curvature in a sphere*, J. Geom. **19** (1982), 197–199.

[**L**] P. L. Lions, *On the existence of positive solutions of semilinear elliptic equations*, SIAM Rev. **24** (1982), 441-467.

[**M**] J. Moser, *On a nonlinear problem in differential geometry*, Dynamical Systems, M. M. Peixoto (ed), Academic Press, New York, 1973, pp. 273–280.

[**R1**] P. H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal. **7** (1971), 487-513.

[**R2**] ——, *On bifurcation from infinity*, J. Differential Equations **14** (1973), 462–475.

[**T**] E. E. L. Turner, *Positive solutions of nonlinear eigenvalue problems*, C.I.M.E. Varenna, 1974, pp. 212–239.

[**Y**] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **12** (1960), 21–37.

[**Z**] S. Zucker, *L_2 cohomology of warped products and arithmetic groups*, Invent. Math. **70** (1982), 169–218.

DEPARTAMENTO DE MATEMÁTICA, FCEYN, UNIVERSIDAD DE BUENOS AIRES, BUENOS AIRES, ARGENTINA

INSTITUTO ARGENTINO DE MATEMÁTICA (CONICET), VIAMONTE 1636, 1055 BUENOS AIRES, ARGENTINA